Arthur König (in collaboration with Conrad Dieterici)

Die Grundempfindungen und ihre Intensitäts-Vertheilung im Spectrum (Fundamental sensations and their intensity distribution in the spectrum)

Sitzungsberichte der Akademie der Wissenschaften in Berlin 29 July 1886, pp. 805-829

An English translation with a short biographical introduction by Rolf G. Kuehni and a retrospective introduction by Claudio Oleari.

Arthur König 1856-1901

Arthur Peter König was born on September 13, 1856 in Krefeld, Germany. His father was a public school teacher. Born with congenital kyphosis his health was always very delicate but his intellectual abilities were excellent. After finishing high school König entered a three-year commercial apprenticeship before he was allowed to study, first in Bonn and Heidelberg, moving to Berlin in the fall of 1879. There he obtained a broad education in mathematics, physics, chemistry and humanities. Among his professors was Hermann von Helmholtz (1821-1895) under whose supervision he wrote his doctoral thesis on galvanic polarization and of mercury and its relationship to surface tension (1882). In the same year he was named one of Helmholtz's assistants. Two years later he advanced to lecturer and in 1889 he was named director of the physical department of the Physiological Institute of the University of Berlin. In the same year he married Laura Köttgen with whom he had a son, Arthur, who became an astronomer. Circulatory problems caused by his kyphosis resulted in his premature death in 1901.[1] He is buried in Berlin.

König was a key supporter of his supervisor and friend Helmholtz in the latter's intellectual battle with Ewald Hering in the later 19th century, supplying him with much solid data concerning the validity of the Young/Maxwell/Helmholtz trichromatic theory. After Helmholtz's death König completed the editing of the second edition of *Handbuch der physiologischen Optik* (1896, Treatise on physiological optics) adding a bibliographical list of nearly 8,000 entries on the subject of vision. He also edited and proofread Helmholtz's *Vorlesungen über theoretische Physik* (Lectures on theoretical physics).

Originally working in physics, König began in 1883 to concentrate on physiological optics where he published over thirty papers, some of seminal importance. Among these is the here translated 1886 paper (together with C. Dietrici) 'Fundamental sensations and their sensitivity in the spectrum', an empirical determination of what in fact is the spectral sensitivity of the human rod and cone sensors of vision. Earlier attempts at such measurements, but based on much simpler technology, had been made in 1860 by the English physicist James Clerk Maxwell (1831-1879). Using newly development spectrophotometric equipment and modifications of the experimental procedure König and Dieterici published a more detailed paper in 1892, determining the "fundamental sensations" not only of subjects with normal color vision (trichromats) but also of di- and monochromats (Fig. 2).[2] With these measurements König provided evidence for the conjecture that the most common form of color blindness, dichromacy, is due to the absence of one cone type in the eye. For about 35 years averaged König functions were widely used in psychophysical color stimulus calculations until new data based on a different method and involving many more observers were determined by J. Guild and W. D. Wright in the later 1920s, resulting in the recommendations of standard observer data by the *Commission Internationale de l'Eclairage* (CIE, International Commission on Illumination) in 1931.

Other important investigations of König involve the sensitivity of the normal eye for differences in wavelength of light,[3] dependence of the Newton/Grassmann laws of color mixture on light intensity,[4] validity of Fechner's law at different light intensities,[5] brightness of spectral hues at different light intensities,[6] and the similarity between the perceptual sensitivity of the rod cells and the absorption spectrum of the rod photopigment, rhodopsin.[7] König's 32 papers on physiological optics were published posthumously in book form in 1903.[8]

König was very active as an editor. In 1889 he became the sole editor of *Verhandlungen der Deutschen Physikalischen Gesellschaft.* From 1891 on, together with the psychologist H. Ebbinghaus, he edited the journal *Zeitschrift für Psychologie und Physiologie der Sinnesorgane*.

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Retrospective Introduction to

Arthur König (in collaboration with Conrad Dieterici)
Fundamental sensations and their intensity distribution in the spectrum

Human knowledge has evolved over time. Therefore when reading a historic scientific paper the reader needs to take into account what was known and what was unknown when the paper was written. Only in this way can the reader appreciate the contribution and the originality of the scientist. The König paper considered here is a milestone in the evolution of color-vision science.

Today, König's fundamentals are explicitly linked to the spectral sensitivities of the human cone photoreceptors, and, without confusion, the word *fundamentals* designates two things: the *cone primaries* or *confusion points* and the *cone spectral sensitivities*.

Today, the tristimulus space with the reference frame defined by the cone primaries is called the *cone activation space*. In this reference frame the tristimulus values are the cone activations and the color-matching functions are the fundamentals or cone spectral sensitivities. This representation is very important for physiologists and is possible thanks to the *reduction principle* and to the spectral sensitivities of dichromats and trichromats, first provided by Arthur König in the paper considered here. Today, this principle says that hereditary dichromatic color vision types are reductions of normal trichromacy. All this renders the König paper written in 1886 historically important and induces me to read it carefully. This paper is the first of two (the second one, of 1892, completes and refines the content of this first one: A. König, Die Grundempfindungen in normalen und anomalen Farbsystemen und ihre Intensitätsvertheilung im Spectrum, Zeitschrift für Psychologie und Physiologie der Sinnesorgane 4 (1892) 241-347).

In this paper, König published the spectral sensitivities of a monochromat, of two kinds of dichromats (today, protanopes and deuteranopes), and of normal and anomalous trichromats, that he measured by the color-matching technique. Surprisingly, these measurements are of very high quality and are not very different from the data of today.

A crucial distinction is made between *elementary sensation functions* (Elementar-empfindungskurven) and *fundamental sensation functions* (Grundempfindungskurven). This terminological distinction, important because it distinguishes König's terminology with respect to the previous one of Donders, continues in the Schrödinger papers [Grundlinien einer Theorie der Farbenmetrik im Tagessehen, II Mitteilung, *Annalen der Physik*, IV Folge, 444-456 (1920)]:

- The *elementary sensations* are the result of "the reduction of the unlimited number of color sensations to the smallest possible number of elementary sensations, whose singular or simultaneous activation in varying intensity and varying ratio allows the generation of all other color sensations". Elementary sensitivity functions represent the spectral "strength of the elementary sensations", that König measured by color-matching technique.
- The color sensations, analyzed in a way "completely free of theoretical assumptions", induced König to define the *fundamental sensations*, which are produced "in the periphery of the optical nerve (i.e., one that cannot be further decomposed by any type of

stimulation)" and are coming before the "physiological process inducing color sensations". As a result one might suppose that the elementary sensations follow the fundamental sensations in the visual process.

This distinction, that today is no longer made, is for me unclear because König's understanding of the physiology of the visual process and his method for obtaining the fundamental curves from the elementary ones are not completely explained.

The first section of the paper is dedicated to the elementary sensation measurement and to the related computation. Probably, König's original attempt was to measure the fundamental sensitivity curves directly, but, on analyzing the measured quantities, König saw significant regularities inducing him to state his reduction principle and consequently define and derive the fundamental sensitivity curves **R**, **G** and **B** from the elementary sensitivity curves *R*, *G* and *V*. The *reduction principle*, given in the second section, regards the eye's physiology of the dichromats related to that of trichromats and states: "*We can consider the two types of dichromatic color systems ... to be the result of missing the fundamental sensation* **R** *in one case and missing the fundamental sensation* **G** *in the other*." Particularly, the comparison of the elementary sensation curves of monochromats, dichromats and trichromats induced König to suppose that the "number of basic sensations must conform to the number of elementary sensations" and, applying the reduction principle, to define the fundamental sensation curves as a "superposition" (i.e. a linear combination) of elementary sensation curves. This holds true for dichromats and trichromats, but not for monochromats. All this is the answer to the question asked by König at the beginning of the second section: the question is "if we can draw conclusions concerning the physiological process inducing color sensations from" the elementary sensation measurements.

Today, the reduction hypothesis is the important key to define the fundamental sensation curves and, consequently, the fundamental reference frame in the tristimulus space.

König made other important remarks, relating psychology to physiology, and inducing me to a comparison and to an indulgent analysis, that could explain the reason why the terminological distinction is today no longer made.

1) König made the distinction between elementary and fundamental sensations because he thought that the color-matching technique used did not produce the fundamental sensitivity curves directly. Today we say that measurement of elementary sensitivity curves, with the complicated (not fully explained and probably incorrect) computational procedures used by König, is not necessary and it is enough to measure the color-matching functions in the laboratory-reference frame RGB (as previously made by Maxwell, whose functions correctly have negative lobes) and make a linear transformation towards the reference frame with axes defined by the dichromat-confusion points (only after this transformation color-matching functions are positive at any wavelength). This linear transformation is possible by applying the reduction principle. The historical value of the König paper is mainly this principle. This induces us to compare the König spectral sensitivities to the cone sensation curves related to the CIE 1931 observer obtained with the Pitt-Wright confusion points. It follows that these cone-sensation curves are closer to the König elementary sensation curves than to the König fundamental sensation curves (see Fig. 1 below).

2) König said that the *basic* color sensations are “identical with the colors designated by Mr. Hering, based on a purely psychological analysis of color sensations, as *primary red* [Ur-Rot], *primary green* [Ur-Grün] and *primary blue* [Ur-Blau]”. But the “spectral light complementary to basic sensation **B**, with an approximate wavelength of 575 μμ, is Mr. Hering’s *primary yellow* [Ur-Gelb] and corresponds to the crossover wavelength of basic sensation curves **R** and **G**”. Therefore, the primary yellow appears to be produced by a different mechanism to that of all the other primary hues. This apparent phenomenon is surprising and requires explanation.
3) Today, as known, psychophysical colorimetry is based on color matching and no observers’ judgment is required. König referred his fundamentals to Hering’s red, green and blue primary hues and this requires judgments by the observer. Any judgment is dependent on the visual adaptation and, in visual processing, is a step after cone activation. Therefore we do not derive the fundamental sensation curves by applying the color equation to the basic colors and this appears clearly in the comparison made in Fig. 1, where the CIE curves are closer to the König dichromat elementary sensation curves than to the fundamental ones.
4) The trichromat fundamental sensations cannot be completely measured, because the constraints given by the color equations are not sufficient, therefore, today, it is not surprising that these curves are a linear combination of the elementary curves. (Moreover, it seems to me that the König computational iterative technique is not given with enough details to try making a check.) In detail, the König results are:

   a) the fundamental sensation curve **B** is equal to the corresponding elementary sensation curve for dichromats and trichromats;
   b) both the fundamental sensation curves of the deuteranope are equal to the corresponding elementary sensation curves;
   c) only the fundamental sensation curve **G** of the protanope is a mixture of his elementary sensation curves;
   d) with regard to the trichromats, both the fundamental sensation curves **R** and **G** are a mixture of the elementary sensation curves *R*, *G* and *V*.

As a result, I consider the König elementary sensation curves to be “fundamentals” for the dichromats (protanopes and deuteranopes) and, together, “fundamentals” for the trichromats and here “fundamental” assumes the meaning of today. Thus, the terms “Grundempfindung” and “Elementarempfindung” are technical terms that can only be understood within the precise theoretical and experimental framework of König and Dieterici. There are three subjects in this paper I find perplexing:

1) the distinction between “Grundempfindung” and “Elementarempfindung” curves is physiologically and perceptively incomplete;
2) the derivation of the “Elementarempfindung” curves only from color-matching equations is impossible without further hypotheses and data;
3) the derivation of the “Grundempfindung” curves from the “Elementarempfindung” curves is insufficiently explained and the way for obtaining the numbers used in this derivation is unknown.

Finally, the instrumental difficulties of that time (1886) render this work worthy of great attention and my attention goes to the instrumental apparatus, that, unfortunately, is not completely described in the paper (the complete description is in the 1892 paper: A. König, Die Grundempfindungen in normalen und anomalen Farbsystemen und ihre Intensitätsvertheilung im Spectrum, Zeitschrift für Psychologie und Physiologie der Sinnesorgane 4 (1892) 241-347). The light source was a gas lamp, the dispersion was by prisms, and the amount of light necessary for color matching was defined by varying the width of entrance slits and the orientation of the Nicol prisms. König said that the "results were then converted to the interference spectrum and afterwards to sunlight". These two operations concern instrument calibration: the wavelength calibration was made by creating a correspondence of his empirical data, obtained by a prism with known refraction index, to a reference spectrum obtained by interference, and the radiometric calibration was based on measurements made with sunlight. This last step was made "with the help of a special photometric measurement" (not described in this paper, but in the 1892 paper). I suspect that the sun's spectrum was supposed be an equal energy spectrum because in the comparison with the CIE 31 observer the maximum of the König **R** curve is shifted towards a longer wavelength and the curve **B** is higher in the shortest wavelength region.

Below is a reproduction of the spectroscope used for mixing two spectral lights, the part of the instrument well described by König in the 1892 paper (Fig. 2). This spectroscope is an evolution of the spectroscope used in an earlier Helmholtz project, presumably the investigation of complementary lights.

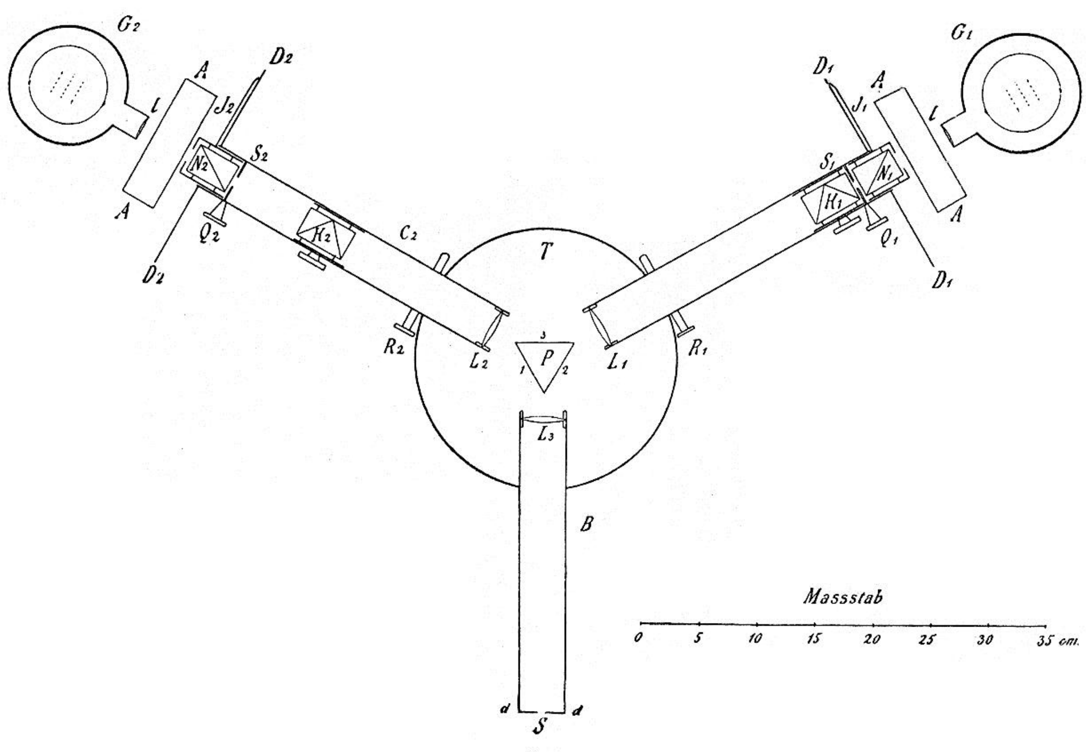

Figure 2.

G_1 and G_2: gas lamps
AA: cutoff infrared light filters
C_1 and *C_2*: two collimators
N_1 and *N_2*: rotating Nicol polarizers
D_1 and *D_2*: discs with angular scales for selecting the plane of polarization of the Nicol polarizers *N_1* and *N_2*
S_1 and *S_2*: entrance slits
K_1 and *K_2*: polarizers
L_1 and *L_2*: lenses with focal lengths *L_1-S_1* and *L_2-S_2*
P: equilateral prism for light dispersion
L_3: lens with focal length *S-L_3*
S: exit slit of the telescope *B*, selecting two parts of two different spectra, that appear mixed to the observer. The selection of the spectral parts is made by changing the directions of the collimators *L_1* and *L_2* with respect to the prism *P*. The intensities of the two spectral parts are modulated by the entrance slits *S_1* and *S_2*, and by selecting the polarization planes of the Nicols *N_1* and *N_2*. The lens *L_3* combined with *L_1* and *L_2* focuses on the plane of the exit slit *S* the entrance slits *S_1* and *S_2*.

Claudio Oleari

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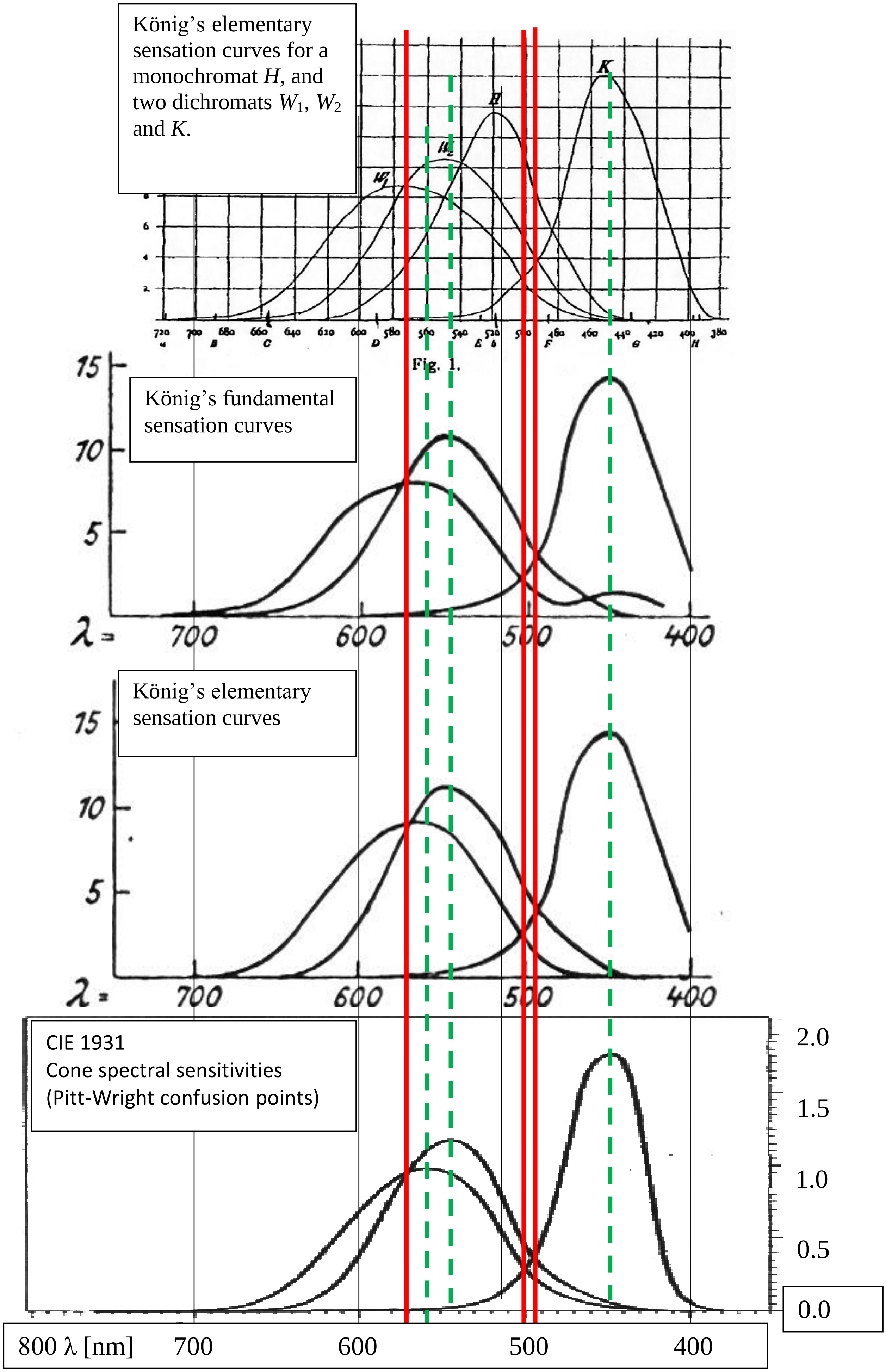
König's elementary sensation curves for a monochromat H, and two dichromats W1, W2 and K.
K
H
W2
W1
720 700 680 660 640 620 600 580 560 540 520 500 480 460 440 420 400 380
B C D E b F G H
Fig. 1.
König's fundamental sensation curves
15
10
5
λ =
700
600
500
400
König's elementary sensation curves
15
10
5
λ =
700
600
500
400
CIE 1931
Cone spectral sensitivities
(Pitt-Wright confusion points)
2.0
1.5
1.0
0.5
0.0
800 λ [nm]
700
600
500
400

Figure I.

Translation

Arthur König (in collaboration with Conrad Dieterici)

Fundamental sensations and their intensity distribution in the spectrum

(Die Grundempfindungen und ihre Intensitäts-Vertheilung im Spectrum)

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I.

Insight into the function of the elements of the visual faculty that sense the light stimulus must have its beginning in the reduction of the unlimited number of sensations of color to the smallest possible number of "elementary sensations" whose singular or simultaneous activation in varying intensity and varying ratio allows the generation of all other sensations of color. This is a task for purely experimental research and it is possible to keep the solution free of any theoretical assumptions, as is done in the following. For the same reason the term "elementary sensations" was selected in distinction to Mr. Donder's decomposition of color systems into "fundamental colors." Mr. Donders, to wit, defines[1] a fundamental color as one representing a simple peripheral process and then identifies it with what we denote with elementary sensations. But this exceeds experience, a situation to be avoided here, all the more because in the course of our investigation a difference between "elementary sensation" and "fundamental color" will become apparent

For *all* color systems the first significant simplification of our task results from the fact that all sensations can be generated with spectral colors and their mixtures so that by reduction of spectral colors to elementary sensations the goal is already achieved.[2] The term to be used for curves that are a result of plotting the strength of the elementary sensations as ordinates, using the interference spectrum as the abscissa axis, is "elementary sensations curves."

a. Monochromatic color systems

Because in this case it is sufficient to assume a *single* elementary sensation, only measurement of the intensity distribution in the spectrum is required to have knowledge of the dependence of stimulation on the kind of stimulus.

Measurements were made[3] using a dispersion spectrum of a uniformly burning gas lamp found to be particularly suitable. Results were next converted to the interference spectrum and then to sunlight.[4] Justification for such conversions came from special experiments where in the dispersion spectrum of the gaslight the intensity ratio between a series of pairs of locations spaced widely apart in the spectrum was repeatedly determined with *varied* slit widths und where the results within a given pair were always found to be identical. This demonstrates that the relationship between the strength of the sensation and the intensity of the light does not change as a function of wavelength.

The table below contains brightness ratios, i.e., the strength of the elementary sensation *H* in the interference spectrum of the sunlight. Here, *as in all subsequent tables*, the unit of measurement for the elementary sensation has been selected in a way that

$$\int H d\lambda = 1000$$

where we have defined *H* as a function of wavelength λ with the unit of the integration variable being 1μμ (μμ = millionth millimeter [nanometer]).

λ	*H*	λ	*H*	λ	*H*	λ	*H*
655	0.006	580	2.376	520	13.772	464	2.312
631	0.045	570	3.989	510	12.801	454	1.097
619	0.133	560	5.684	500	10.765	448	0.446
610	0.392	550	8.025	490	6.737	437	0.115
600	0.836	540	10.093	480	5.290	426	0.070
590	1.345	530	12.016	474	3.239		

The progress of curve *H* is plotted in Fig. 1. So far only Mr. Donders has made identical determinations for a single monochromatic color system.[5] His results, to the extent one can conclude from the published data, are in complete agreement with the above results, so that there is at least some justification to consider the observed case as typical.

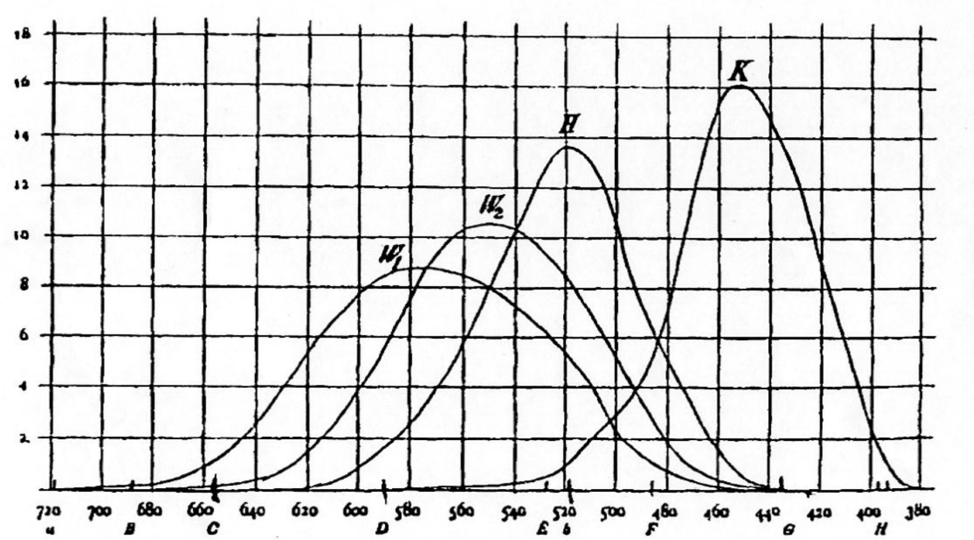

Fig. 1.

b. Dichromatic color systems

In dichromatic color systems all of the spectral colors can be produced through mixture of lights at both ends of the spectrum. It, therefore, is sufficient for the complete analysis of a system to assume the sensations generated by the latter to be elementary sensations.

Experience has taught that these elementary sensations, in line with Mr. Donder's proposal we want to designate them as *warm* and *cold*, are not only induced by the extreme ends of the spectrum but that there is a region at both ends in which only the intensity of color changes. We want to designate these two parts of the spectrum as "end sections" and the part that is enclosed by them as "middle section."

The simplest path for determining the curves of the elementary sensations in this situation is as follows.[6] If we designate with L the amount of light belonging to equally wide sectors of the spectrum and with W and K the two elementary sensations therein and apply indices 1 and 3 to two segments of the spectrum located in the end sections and index 2 to one in the middle section a color equation can be written as follows

$$L_2 = aL_1 + bL_3$$

where a and b designate two coefficients to be determined experimentally.

This and all following color equations have been, where possible, established repeatedly at the same intensity until the probable error for the coefficients a and b was less than 1% of their value.

That such color equations do not depend on absolute intensity was repeatedly subjected to careful testing and was found to be valid for the four dichromatic color systems described below.[7]

In two colors of equal appearance each elementary sensation must be present in equal strength and as a result in the color equation we can replace L with W as well as with K.

Because $W_3 = K_1 = o$

we obtain $W_2 = aW_1$

and $K_2 = bK_3$.

The position of segment 2 is one of choice. Therefore, it is possible to determine values for W and K for any desired segment in the middle section. Initially the unit of measure has to be set arbitrarily. The shape of the elementary sensation curves in the two end sections has to be determined (as in a monochromatic system) by comparing intensities.

In practice this method suffers from the drawback that because of the large distance between the spectral segments with indices 1 and 3 the numerical values of the coefficients a and b cannot be determined with desirable accuracy. For this reason the following theoretically more complex but practically more productive method has been used for three of the investigated dichromatic systems. L, W and K have the same meaning as above; indices 1 and 7 now refer to the end sections, 2 to 6 to the middle section. The following color equations were formed

$$L_2 = a_2L_1 + b_2L_5 \qquad 1)$$

$$L_3 = a_3L_1 + b_3L_5 \qquad 2)$$

$$L_4 = a_4 L_1 + b_4 L_5 \quad 3)$$
$$L_4 = a'_4 L_3 + b'_4 L_7 \quad 4)$$
$$L_5 = a_5 L_3 + b_5 L_7 \quad 5)$$
$$L_6 = a_6 L_3 + b_5 L_7 \quad 6)$$

When L is replaced by W and one takes into account that $W_7 = 0$ the following is obtained from equations 4), 5) and 6)

$$W_4 = a'_4 W_3 \quad 7)$$
$$W_5 = a_5 W_3 \quad 8)$$
$$W_6 = a_6 W_3 \quad 9)$$

If in equations 2) and 3) L is replaced by W and equations 7) and 8) are employed three different expressions for W_1 can be derived, that is,

$$W_1 = \frac{a'_4 - b_4 a_5}{a_4} W_3$$
$$W_1 = \frac{1 - b_3 a_5}{a_3} W_3$$
$$W_1 = \frac{b_4 - a'_4 b_3}{a_3 b_4 - a_4 b_3} W_3$$

If coefficients a and b are determined with perfect accuracy the three values of W_1 should be numerically identical. Because of errors in observation this is unlikely to be fully the case.

Color equations involving light of the same wavelength in very different intensities have been used. The fact that the deviations have been minimal provides the best proof that color equations do not depend on absolute intensity. The value of W_2 was calculated from equation 1) by using the average of the three never much varying individual values of W_1. In the end section, containing the segment designated by index 1, the shape of the elementary sensation curve W (still in units of the arbitrarily selected value of W_3) was obtained by intensity comparison, as in the first mentioned method.

The determination of the second elementary sensation curve K was achieved in completely analogous manner.

In the actual measurements more than five positions in the middle section were considered in the color equations to determine the exact shape of the curve. As a result we were often in a position to deduce the presence of sources of error from lack of smoothness of curve and to eliminate them.

The two elementary sensation curves obtained in this fashion were related to the dispersion spectrum of the gas flame followed by conversion to the interference spectrum of sunlight in the same manner and with the same justification used for the monochromatic color system. The initially arbitrary scale of the ordinate was then also converted in a way that, with the earlier established supposition for the unit of length, the area between the curve and the abscissa axis had a value of 1000.

It must be kept in mind that to equate the two areas, i.e., the strength of activation of the two elementary sensations by sunlight, is a purely mathematical operation. A numerically expressed quantitative relationship of the two different elementary sensations is not possible. Establishing measuring units in this way can be justified in the same manner for any other light, e.g., for the light of a gas lamp.

When making such a conversion for light of any kind the abscissa of the crossing point of the two elementary sensation curves denotes wavelength λ_n of the spectral light that, for the person endowed with the related dichromatic color system, creates the same impression as the monochromatic light and for which, as a result, equation

$$\frac{W_{\lambda n}}{\int W d\lambda} = \frac{K_{\lambda n}}{\int K d\lambda}$$

applies.

For the investigated dichromatic systems the approximate agreement[8] of the wavelength of this crossing point, determined by calculation and plotting for gas as well as sunlight, with the wavelength determined in direct observation (comparison of unrefracted with monochromatic light) is taken as confirmation of the correctness of the obtained elementary sensation curves.

The following tables contain the values of *W* and *K* relative to the solar interference spectrum for four dichromatic color systems.[9]

In regard to these tables it should be noted that the values in parentheses, applicable in all cases to the most extreme, dark areas of the end sections, were obtained by a procedure that is not completely exact and the description of and justification for which is postponed to a more detailed presentation. The values involved are very small and therefore do not have a noticeable influence on the total shape and the scale of the curves.

Obs.	W.W.		E.B		L.K.		H.S	
λ	W_1	K	W_1	K	W_2	K	W_2	K
720	(0.029)	-	0.031	-	(0.002)	-	0.004	-
700	(0.099)	-	0.100	-	(0.006)	-	0.013	-
685	(0.204)	-	0.208	-	(0.012)	-	0.029	-
670	0.471	-	0.480	-	0.027	-	0.065	-
660	-	-	-	-	0.051	-	0.345	-
650	1.610	-	0.799	-	-	-	-	-
645	-	-	-	-	0.192	-	-	-
642	2.398	-	-	-	-	-	-	-
632	-	-	-	-	0.414	-	-	-
630	4.045	-	-	-	-	-	1.026	-
620	5.600	0.001	5.122	0.005	0.919	-	-	-

610	-	-	-	-	2.367	-	2.735	-
605	7.234	0.029	6.891	0.030	-	-	-	-
600	-	-	-	-	3.703	-	3.854	-
590	8.244	0.038	8.385	0.057	5.418	-	5.708	0.003
580	-	-	-	-	7.043	-	7.639	0.012
575	-	-	8.716	0.068	-	-	-	-
570	8.567	0.110	-	-	8.784	-	10.016	0.020
560	-	-	8.594	0.104	9.798	-	-	-
556	-	-	-	-	-	-	10.817	0.091
550	7.852	0.212	-	-	10.225	-	-	-
545	-	-	7.932	0.178	-	-	-	-
540	-	-	-	-	-	-	10.423	0.259
535	-	-	6.971	-	9.901	-	-	-
530	6.090	0.615	-	0.409	-	-	-	-
525	-	-	-	-	-	-	8.914	0.622
521	-	-	-	-	8.806	6.616	-	-
515	-	-	4.608	1.228	-	-	-	-
510	4.784	1.475	-	-	-	-	6.867	1.436
503	-	-	-	-	6.555	1.912	-	-
500	2.392	2.552	2.562	2.809	-	-	4.163	2.321
487	0.996	4.707	1.319	5.988	4.226	5.216	2.074	4.290
479	-	-	-	-	1.643	9.054	-	-
475	0.596	10.348	0.656	10.920	-	-	1.251	8.324
467	-	-	-	-	0.045	14.205	-	-
465	0.348	12.903	0.025	13.776	-	-	0.736	12.892
455	0.157	14.786	-	-	-	18.007	0.347	15.004
445	-	-	-	-	-	-	-	12.262
440	0.000	14.142	-	-	-	13.980	-	-
439	-	-	-	-	-	-	-	15.600
438	-	-	-	12.605	-	-	-	-
436	-	-	-	-	-	13.056	-	-
430	-	-	-	-	-	10.826	-	-
420	-	-	-	-	-	4.906	-	-
400	-	(2,343)	-	(2.018)	-	(2.425)	-	(2.585)

When plotting the eight elementary sensation curves it is immediately apparent that the four curves *K*, aside from small individual deviations resulting from observational errors, are identical while there are two forms of curve *W*; the *W* curves of observers W.W. and E.B with a maximum near 570 μμ belong to the first, those of observers L.K. and H.S with a maximum near 555 to 550 μμ to the second form. Measurements of somewhat reduced accuracy of several additional dichromatic color systems always indicated membership in one of these two forms, so that they must be seen as typical, all the more, because also when using other experimental techniques a separation of all dichromatic systems into two groups corresponding to the above groups is necessary.

We want to distinguish the two types of curve *W* (as already done in the column headings of the table) with added indices 1 and 2. Aside from the already mentioned curve *H* Figure 1 contains average values of curves W_1, W_2, and *K*. In some areas the individual differences are so small that, given the size of the figure, they would not be visible.

As a result of the stronger effect of *individual* differences in the *W* curves, caused by absorption in the macula, compared to differences of *type* the position of the neutral point in the spectrum cannot be included among the valid indicators of difference between the two types.[10]

Mr. Donders, without directly contradicting experience, identifies his fundamental colors in the dichromatic color systems with what we here call elementary sensations; the curves from the above cited articles by Mr. van der Weyde, showing intensity of fundamental colors in dichromatic color systems, indicate complete agreement of the curves identified as "cold fundamental color" with our curves *K*. However, his two curves of "warm fundamental colors" deviate from our curves W_1 and W_2 in that their maxima are shifted in the direction of the short-wave end of the spectrum. The differences are of such a nature that they can be explained to a lesser degree with errors of observation but probably to a larger degree with a difference in the composition of the light source.[11] This situation has almost no effect in case of the narrow shape of curves *K*.

Aside from the described two types of dichromatic system another form of so-called color blindness has been mentioned, "violet," respectively "blue-yellow," blindness. However, so far no accurate measurements of this type exist.[12]

c. Trichromatic color systems

Analysis of trichromatic systems requires the supposition of three elementary sensations and is particularly difficult because only two of these are present in full saturation in the spectrum while the third can never be sensed purely but only in reduced saturation.

As in case of dichromatic color systems it is apparent that at the ends of the spectrum, over a relatively extended range, color changes *only* in intensity. We, again, want to designate these two regions as "end sections" and select the sensations induced by them, i.e., spectral red and violet, as two of the three required elementary sensations. They are designated *R* and *V*.

Adjacent to the end sections in both cases, in direction of the center of the spectrum, is a region in which each hue can be generated by mixture of light of the end section with that of the spectral color located at the inner border. They are, in a certain sense, dichromatic regions and we want to call them "intermediate sections." Here the third elementary sensation designated with *G* is added to the pure elementary sensation of either end section, so that in the first intermediate section elementary sensations *R* and *G* are present, in the second *G* and *V*. In the region enclosed between the two intermediate sections, the "middle section," all three elementary sensations are engaged.

The empirical fact that *no* light of an intermediate section can be matched with lights from the two end sections is an indication that the elementary sensation added in the middle section to that of an end section cannot be that of the other end section. It, therefore, must be an elementary sensation *different from those* of the two end sections and in both intermediate sections *the same one* because, otherwise, we would have in total four elementary sensations. Their existence would conform to a color system of fourfold diversity that would make it possible to have aside from the attributes intensity, hue and saturation a fourth one, a situation that is in contradiction to experience.

Based on our observations the borders of these sections are, with very small individual differences, the following:[13]

First end section	extreme red - 655 μμ
First intermediate section	655 μμ - 630 μμ
Middle section	630 μμ - 475 μμ
Second intermediate section	475 μμ - 430 μμ
Second end section	430 μμ - extreme violet

It should be emphasized that the border between the first intermediate and the middle section (630 μμ) and that between the second intermediate and the second end section (430 μμ) can only be determined with limited accuracy, the former because of the insensitivity of the eye in regard to small saturation differences in this area of the spectrum, the latter because of low intensity at the short-wave end of the dispersion spectrum of the lamp in use.

The first of these circumstances was very troublesome insofar as it required us to determine elemental curve *V* according to a significantly different method.

Lord Rayleigh[14] and Mr. Donders[15] have demonstrated that trichromatic color systems vary significantly and can be sorted, without evident transition, into at least two groups. The first group has the largest number of members while the second so far clearly distinguished group is found not more frequently than are dichromatic systems.[16]

1. Normal trichromatic color systems

The authors of this article were the observers used for the determination of the trichromatic elementary sensation curves.

Suitable color mixtures were very difficult to find and success was only obtained after multiple failed trials. Only color mixtures of the kind may be made where equality of the resulting colors according to hue and saturation can be sensitively determined and where the observational error of the combination does not have a large effect on the result of the numerical calculation. In consideration of the first matter whitish colors must be avoided, therefore generally only parts of the spectrum that are located close to each other must be mixed. On the other hand, calculation accuracy makes it desirable that the two components of a mixture are as separated from each other as far as possible. The satisfactory results reported below could only be achieved with careful balancing of these two constraints in case of each mixture.

More detailed information concerning individual mixtures that we used must await a more exhaustive presentation. Only the following should be mentioned here. The curves of the elementary sensations *R* and *G* were determined according to the second method described for dichromatic systems. At the start the course of the rising curve *G* in the first intermediate section was determined and then, with a procedure completely analogous to the one used in the determination of the value of the ordinate W_1, a value of the ordinate *G* of the middle section was calculated. With the help of the curve segment made known in this manner we advanced in steps until the border between the second intermediate section and the second end section was reached. Because in this procedure small saturation differences often had to be evened out we had to introduce into the calculations ordinates of *G* belonging to an initially not calculated but, based on preliminary tests, only approximately known portion of the curve. After calculations were completed as far as the given border point it was possible to obtain, either directly or with help from graphical interpolation, improved values for these rather small adjustment figures and thereby to calculate curve *G* in second approximation. This process was continued until an additional recalculation resulted in no further change in the curve, i.e., until the curve was internally consistent and thereby *uniquely* determined. The best proof for the accuracy of all used mixtures was the fact that the last ordinate of *G* with a theoretical value of zero had, as we have already seen above, an extremely small actual value.

Calculation of the elementary sensation curve *R* was begun in comparable manner on the border of the second intermediate section and the middle section and was continued to some point in the first end section where its further course was found with the help of determination of intensity ratios. Also here, for reasons identical to those given for curve *G*, calculations had to be carried out several times.

As a result, the two elementary sensation curves *R* and *G* were based on the same justification and determined according to the same method given for dichromatic color systems, recalculated for the interference spectrum with scale reductions for the lamp light such that (as before for sunlight)

$$\int R d\lambda = \int G d\lambda = 1000 .$$

The wavelength of the crossover point of the curves reduced in this fashion is designated with λ_{rg}. As a result

$$\frac{R_{\lambda rg}}{\int R d\lambda} = \frac{G_{\lambda rg}}{\int G d\lambda} .$$

If we label the wavelengths of a pair of spectral colors that can be mixed to have the same color as the non-refracted lamplight with λ_1 and λ_2 and with *c* a factor dependent only on these two wavelengths, then if *R, G* and *V* are expressed with the scale

$$\int R d\lambda = \int G d\lambda = \int V d\lambda$$

each pair conforms to the equation

$$R_{\lambda 1} + cR_{\lambda 2} = G_{\lambda 1} + cG_{\lambda 2} = V_{\lambda 1} + cV_{\lambda 2}.$$

If we now identify λ_1 with λ_{rg} it follows, because $R_{\lambda 1} = G_{\lambda 1}$, from the first half of the last equation that

$R_{\lambda 2} = G_{\lambda 2}$.

Experience has taught us that there is only one crossover point between the elementary sensation curves *R* and *G*. As a result, for $\lambda_1 = \lambda_{rg}$,

$R_{\lambda 2} = G_{\lambda 2} = 0$,

i.e., only light of the second end section can be mixed with light of the crossover point λ_{rg} to obtain the color of the light of our lamp. We obtained

	From curves	Experimentally	Difference
For *K*	λ_{rg} = 589.8	588.8	-1.0
For *D*	λ_{rg} = 586.0	585.5	-0.5

It follows from very similar considerations that the spectral light completing the light of the first end section to result in the non-refracted lamplight is identical with the light of the crossover wavelength of *G* and *V*. We designate it with λ_{gv}.

The experiment resulted in the values

for *K* λ_{gv} = 516.5 μμ

for *D* λ_{gv} = 512.0 μμ.

The theoretically most obvious method for *determination of the elementryl sensation curve V* is to combine light of the border between the first intermediate section and the middle section with a light located in the middle section and then to determine the progress of curve *V* from there to the short-wave end of the spectrum in a fashion similar to how we determined curve *R,* moving from wavelength 475 μμ in direction of the long-wave end of the spectrum. Insensitivity of the trichromatic eye to saturation differences for hues in the middle section prohibited the use of this method and was the cause for devising a procedure based on knowledge of the wavelength of λ_{gv}. With values of *G* known and $G_{\lambda gv} = V_{\lambda gv}$ we also know $V_{\lambda gv}$. With help of an initially arbitrary value for *V* at a position located further in direction of the violet end of the spectrum we determined the progress of curve *V* from color mixtures until the beginning of the second end section and in the second end section the curve was plotted on basis of approximating tests. In case of lamplight the spectrum can be ended at 400 μμ without committing a serious error. In the middle section where, for reasons already noted, mixtures did not provide a secure basis for curve *V*, we determined it in a manner that the law of color mixture, according to which a mixture is never more saturated than any spectral color, was obeyed at all locations. As we found in our experimental work, this method of curve determination has an uncertainty of such small magnitude that it has, after recalculation to the interference spectrum, no noticeable influence on the value of $\int V d\lambda$.

The so far completely arbitrary selection of the ordinate of *V* now was varied until the equation

$$\int V d\lambda = 1000$$

was satisfied with the result that the complete progress of the elementary sensation curve *V* was *uniquely* determined. The values *R, G* and *V* were then, comparably to the dichromatic systems, recalculated for sunlight. But because here one cannot, without contradicting practical experience, set at wavelength 400 µµ *V* = 0 the decline of curve V in the second end section was calculated from Fraunhofer's determination of the intensity distribution in the solar spectrum.[17] Thereupon ordinate scale reduction was performed in known fashion.

The following table lists the results, with the approximated values of *V* in parentheses.

λ	For K			For D		
	R	G	V	R	G	V
720	0.033	-	-	0.033	-	-
700	0.110	-	-	0.104	-	-
685	0.231	-	-	0.232	-	-
670	0.519	-	-	0.502	-	-
660	0.905	-	-	0.852	-	-
645	2.170	0.124	-	1.891	0.071	-
630	3.988	0.534	-	3.481	0.339	-
620	5.227	1.106	(0.001)	4.827	0.755	(0.001)
610	6.704	2.168	(0.006)	6.246	1.648	(0.006)
600	7.400	3.711	(0.016)	7.076	2.880	(0.016)
590	8.326	5.541	(0.034)	7.988	4.635	(0.034)
577	8.965	8.275	(0.079)	8.799	7.430	(0.067)
563.5	9.505	11.011	(0.169)	9.100	9.911	(0.168)
555	9.471	11.782	(0.260)	9.095	10.858	(0.259)
545	8.776	11.933	(0.394)	8.557	11.217	(0.392)
536	7.709	11.070	0.608	7.857	10.718	0.564
516.5	4.081	7.338	1.247	-	-	-
512	-	-	-	4.158	8.016	1.469
505	2.174	4.473	1.811	3.134	6.376	2.187
495	1.078	2.610	2.729	1.813	4.296	3.283
485	0.587	2.015	5.629	0.925	3.107	5.280
475	0.000	1.703	10.469	0.000	2.497	10.182
463	-	0.925	13.075	-	1.393	13.401
455	-	0.457	13.421	-	0.810	14.143
445	-	0.123	13.693	-	0.256	14.250
435	-	0.000	12.323	-	0.000	11.900
400	-	-	(2.760)	-	-	(2.674)

The deviations between the curves for *K* and *D* are insignificant. But because their systematic distribution is obvious it follows that only the smallest portion is caused by observational error.

Because equation 1 above applies as well to these curves (related to sunlight), λ_{rg} and the second end section, as well as λ_{gv} and the first, also must be complementary, i.e., when mixed each of these pairs of spectral colors must result in the color of sunlight. The values of λ_{rg} and λ_{gv} have been experimentally determined and the result compared with the graphic plot of the elementary sensation curves. The following was found

	λ_{rg} from curves	λ_{rg} direct observation	difference	λ_{gv} from curves	λ_{gv} direct observation	difference
for *K*	572.8	573.0	+0.2	495.5	496.3	+0.8
for *D*	569.5	570.6	+1.1	491.8	494.1	+2.3

Considering the many experimentally determined factors that entered into the numerical calculation the differences can be considered small so that it is undoubtedly justified to consider the degree of agreement as proof for the validity of the obtained elementary sensation curves.

Figure 2 contains the *R, G* and *V* curves for *D* as well as the *G* curve for *K*. For *K* the other two curves deviate so little from those of *D* that it would be difficult to distinguish them in the

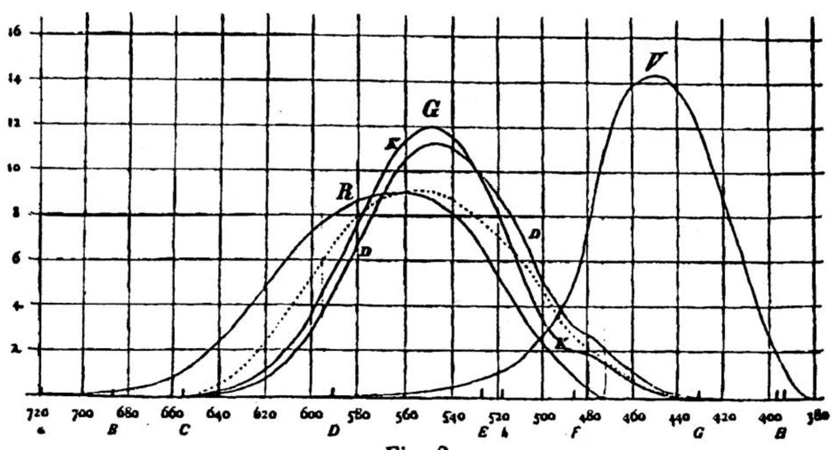

Fig. 2.

graph. – Absorption in the macula lutea is clearly noticeable as a segment disrupting the smooth progress of the curves of *K*, particularly in the plotted curve *G*. This section ranges from approximately 535 μμ to 475 μμ, in agreement with direct observation. For *D* this absorption is much smaller and the interval shorter.

Maxwell[18] and Mr. Donders[19] have subjected normal trichromatic color systems, in two cases each, to experimental analysis also and their results are in essential agreement with ours. Deviations in methodology and their influence on the results cannot be discussed here.

2. Anomalous trichromatic systems

It is still an open question if there are also different groups of anomalous trichromatic color systems and it can only be resolved with much more experimental data. Investigations of many individuals by Mr. Donders has already made it possible to delineate *one* group with good certainty.

We were fortunate to be able to find two representatives of this group,[20] Prof. B. and Engineer Z., who volunteered to have their color system investigated. External circumstances only allowed determination of a small number of color mixtures for Mr.
B while Mr. Z. was at our disposal for longer time periods. In both cases the resulting curves have considerably less accuracy than what we can claim for the determination of our own, normal trichromatic color systems.

The selection of color mixtures and the method of calculation were, in principle, the same as in case of normal systems; however, because much fewer experimental data were obtained the methodology had to be significantly simplified. Deviations in technique will not be discussed here in detail.

The borders of the various above discussed sections of the spectrum were not demonstrably different from those of the normal trichromatic systems. In the present cases we designate the elementary sensation with *R', G'* and *V'*.

Concerning the result of the observations it should be noted, to the extent they relate to the interference spectrum of lamplight, that on basis of the graphical plot the value of λ_{rg} was found to be 600 μμ and the wavelength of the spectral light which, when mixed with light of the second end section, matches the color of our lamplight was found to be 601 μμ. Also here the two values indicate satisfactory agreement. The following table contains, related to sunlight, complete elementary sensation curves of Mr. Z. while those of Mr. B. are only supplied to the extent that they could be accurately calculated by assuming end ordinates to have the same magnitude as in case of Mr. Z. Curve *G*' of Mr. Z. has already been plotted with a dashed line in Fig. 2.

We conclude from the table below:

1. Curve *R*' deviates somewhat from curve *R*. -- Here it should not be kept a secret that a critical review of the dependence of curve shape on uncertainty of observation and calculation results in a form considerably different from that falling within the borders of possible observational error. H*owever, the most important characteristic property of the curve is completely independent of such uncertainty*, as will be further discussed in section II
2. Curve *G*' displays large deviations compared to the normal form. While the curve maximum is located at the same wavelength, the curve type is much different.
3. The deviations between anomalous curve *V*' and normal curve *V*, rather considerable in the interval from 455 to 430 μμ, derive *undoubtedly* from observational error. Here they are particularly large because of the recalculation relative to the interference spectrum

of sunlight. *Before* each recalculation, i.e., when the curves still represent the dispersion spectrum of lamplight, the differences are very small.

λ	Z			B		
	R’	G’	V’	R’	G’	V’
720	(0.044)	-	-			
700	(0.144)	-	-			
685	(0.298)	-	-			
670	0.689	-	-	0.698	-	-
645	2.481	0.291	-	2.555	0.319	-
630	4.020	1.259	-	4.148	1.205	-
620	5.287	2.269	(0.001)	5.349	2.288	-
610	6.690	3.804	(0.004)	7.033	3.826	-
600	7.672	5.250	(0.013)	7.736	5.149	-
590	8.571	6.678	(0.026)	8.140	6.750	-
577	8.678	7.684	(0.041)	8.634	8.252	-
560	8.341	8.964	(0.086)	8.557	9.364	-
545	7.536	8.956	(0.146)	-	-	-
535	6.348	8.274	(0.198)	6.348	7.850	-
520	5.147	7.135	(0.331)	-	7.135	-
510	-	-	-	-	-	0.565
505	4.191	5.958	0.882	-	-	-
495	1.929	3.558	3.129	-	-	3.116
485	1.041	3.288	6.210	-	-	6.274
475	0.000	3.081	10.194	-	-	9.748
463	-	1.784	12.931	-	-	11.154
455	-	0.507	13.971	-	-	13.280
445	-	0.223	13.280	-	-	-
430	-	0.000	13.570	-	-	13.760
400	-	-	(2.985)	-	-	(3.000)

II.

In section I the analysis of sensations of color has been carried out completely free of theoretical assumptions. The question now arises if we can draw conclusions concerning the physiological process inducing sensations of color from what we have obtained so far. From now on we want to define “fundamental sensation” as a sensation corresponding to a simple process in the periphery of the optical nerve (i.e., one that cannot be further decomposed by any type of stimulation).[21] In each color system the number of fundamental sensations must conform to the number of elementary sensations.

Fundamental sensations can be represented as functions of wavelength in the same manner as elementary sensations. We will select the scale again in a manner that the integral over the complete expanse of the spectrum equals 1000. We now introduce the following designations for the fundamental sensations:

For monochromatic systems **H,**
for dichromatic systems $\mathbf{W}_1$ and $\mathbf{K}_1$, respectively $\mathbf{W}_2$ and $\mathbf{K}_2$,
for normal trichromatic systems **R**, **G** and **B**,
for anomalous trichromatic systems **R**′, **G**′ and **B**′.

Because colors that have the same appearance activate the fundamental sensations always to the same extent we can replace *L* in all of our color equations listed in section I with one of the fundamental sensations. Because *L* can also be replaced with an elementary sensation and all color equations are homogeneous and linear the following relations result (except for a here not significant multiplicative constant):

1. for monochromatic systems
   H = *H*;
2. for dichromatic systems
   a) of the first type

$$\mathbf{W}_1 = \frac{\alpha'_1 W_1 + \beta'_1 K}{\alpha'_1 + \beta'_1},$$

$$\mathbf{K}_1 = \frac{\alpha''_1 W_1 + \beta''_1 K}{\alpha''_1 + \beta''_1},$$

   b) of the second type

$$\mathbf{W}_2 = \frac{\alpha'_2 W_1 + \beta'_2 K}{\alpha'_2 + \beta'_2},$$

$$\mathbf{K}_2 = \frac{\alpha''_2 W_1 + \beta''_2 K}{\alpha''_2 + \beta''_2};$$

3. for normal trichromatic systems

$$\mathbf{R} = \frac{a'R + b'G + c'V}{a' + b' + c'},$$

$$\mathbf{G} = \frac{a''R + b''G + c''V}{a'' + b'' + c''},$$

$$\mathbf{B} = \frac{a'''R + b'''G + c'''V}{a''' + b''' + c'''}.$$

Equations of the same form apply in case of anomalous trichromatic systems. Here apostrophes are added to **R**, **G**, **B** and *R, G, V*. We use the term “superposition” to designate the combination of elementary sensation curves corresponding to these equations.

The simplest relationship between the different color systems we can think of involves the assumption that monochromatic and dichromatic systems contain one, respectively two, of the three fundamental sensations of the normal trichromatic system.

For *monochromatic systems* it is evident from calculation (as well as inspection of the curve) that it is impossible to create the elementary sensation curve *H* by any kind of superposition. *The monochromatic color systems so far investigated in detail, therefore, cannot be considered to have come into existence through absence of one or two of the fundamental sensations of the so far investigated di- or trichromatic systems*.

Because of the properties of the visual sense that always exist simultaneously one has to view, with Mr. Donders,[22] monochromatic systems as a pathological abnormality. It is, therefore, not of further concern that there is a lack of a simple relationship to color systems that are not pathologically changed

The result is much different in the matter of the *relationship between dichromatic and normal trichromatic systems*. – Using average values for the elementary sensation curves we find, with entirely sufficient accuracy considering the observational error in the data,

$$\mathbf{W}_1 = \mathbf{R}$$
$$\mathbf{W}_2 = \mathbf{G}$$

and $$\mathbf{K}_1 = \mathbf{K}_2 = \mathbf{B}$$

assuming that

$$\mathbf{W}_1 = \frac{W + 0.1K_1}{1.1}$$
$$\mathbf{W}_2 = W_2$$
$$\mathbf{K}_1 = \mathbf{K}_2 = K$$
$$\mathbf{R} = \frac{R - 0.15G + 0.1V}{0.95}$$
$$\mathbf{G} = \frac{0.25R + G}{1.25}$$

and $\mathbf{B} = V$ (Ref.23).

It must be clearly emphasized here (the proof to be provided in a more detailed presentation) that the *possibility* of the above equalities is *never* threatened by uncertainty of observation. Only the numerical values of the superposition coefficients are affected by it.

We can therefore state: *We can consider the two types of dichromatic color systems so far investigated in greater detail to be the result of missing fundamental sensation* **R** *in one case and missing fundamental sensation* **G** *in the other*.

The tables below contain the result of calculations for average curves $\mathbf{W}_1$ and $\mathbf{W}_2$ of the dichromatic and the individual curves **R** and **G** of the normal trichromatic systems. The last column relates to anomalous trichromatic systems and will be considered later.

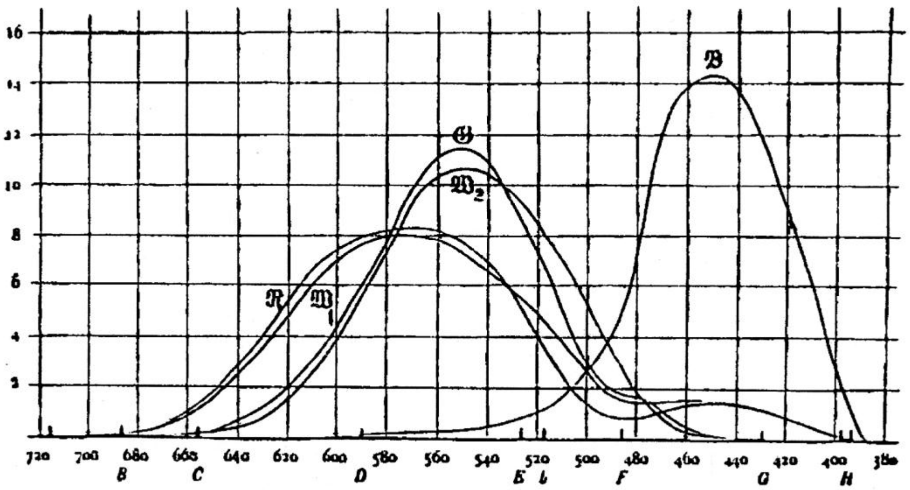

Fig. 3.

The degree of similarity between the related curves is for one of us (D.) such that at the scale of drawing in Fig. 3 the curves coincide for a very short segment. For this reason we have, in addition to curves $\mathbf{W}_1$ and $\mathbf{W}_2$, plotted curves **R** and **G** only for the other author (K.). Here deviations disappear as well by approximately evening out the mentioned discrepancy resulting from light absorption in the macula lutea, as well as appropriate adjustment of ordinate values. Curve **B** is also plotted in the figure.

A more profound view opens when considering also *anomalous trichromatic systems*. The following equations can be written

$$\mathbf{R} = \mathbf{R}' = \frac{R'+0.1V'}{1.1}$$

$$\mathbf{B} = \mathbf{B}' = V',$$

i.e., two of the fundamental sensation curves of *normal* trichromatic systems can be formed by superposition of *R*′ and *V*′.[24] On the other hand calculation (and visual examination) shows that any superposition of *R*′ and *G*′ results in a curve with a form that is *transitional between* **R** *and* **G**. We can view the group of anomalous trichromatic systems investigated by us as the connecting link between normal trichromatic and dichromatic systems of the first type if we assume that in case of the latter the intensity curve of fundamental sensation **G** is moved to coincide with the unchanged curve of fundamental sensation **R**.

Full justification for this assumption derives from hue determination of the three fundamental sensations as well as observations by unilaterally "color blind" observers. The color diagram constructed on basis of curves **R**, **G** and **B** yields the hues that correspond to the fundamental sensations (they can, to some extent, also be directly deduced from Fig. 3):

for **R** a *red* deviating somewhat from the red of the first end section of the spectrum in the direction of purple,

for **G** a *green* of approximate wavelength 505 μμ,

for **B** a *blue* of approximate wavelength 470 μμ.

In addition, the configuration of the color diagram discloses that of the fundamental sensation **B** has the highest and **G** the lowest saturation in the spectrum; furthermore, the color diagram is in agreement with the empirical fact that spectral violet always has higher saturation than any mixture of spectral blue with spectral red.

For dichromatic systems			For *K*		For *D*		For *Z*
λ	**W**$_1$	**W**$_2$	**R**	**G**	**R**	**G**	**R'**
720	0.026	0.003	0.034	0.006	0.034	0.006	0.033
700	0.087	0.010	0.116	0.021	0.109	0.020	0.130
685	0.176	0.020	0.234	0.043	0.244	0.043	0.270
670	0.437	0.046	0.546	0.100	0.529	0.100	0.627
650	1.43	0.233	-	-	-	-	-
645	-	-	2.264	0.533	1.979	0.435	2.265
630	3.55	0.76	4.112	1.234	3.610	0.967	3.565
620	4.92	1.48	5.327	1.930	4.962	1.570	4.806
610	6.04	1.55	6.714	3.075	6.316	2.568	6.082
610	7.00	3.78	7.205	4.449	7.000	3.719	6.975
590	7.64	5.56	7.892	6.097	7.680	5.306	7.800
580	7.97	7.34	-	-	-	-	-
577	-	-	8.139	8.713	8.110	7.704	7.893
570	7.99	9.40	-	-	-	-	-
563.5	-	-	8.284	10.709	8.042	7.749	-
560	7.77	10.27	-	-	-	-	7.591
555	-	-	8.137	11.320	7.886	10.507	-
550	7.37	10.55	-	-	-	-	-
545	-	-	7.395	11.300	7.278	10.685	6.865
540	-	10.39	-	-	-	-	-
536	-	-	6.432	10.398	6.637	10.146	5.790
530	5.80	9.64	-	-	-	-	-
520	5.00	8.50	-	-	-	-	4.711
516.5	-	-	3.269	6.686	-	-	-
512	-	-	-	-	3.266	7.244	-
505	3.31	6.26	1.772	4.014	2.523	5.727	3.890
495	2.03	4.31	1.101	2.303	1.576	3.800	2.038
485	1.50	2.72	0.892	1.730	1.040	2.670	1.511
475	1.41	1.265	0.834	1.362	0.678	2.000	0.927
463	1.44	0.520	1.230	0.740	1.201	1.114	1.165
455	1.45	0.173	1.340	0.366	1.360	0.648	1.179
445	1.37	-	1.407	0.170	1.460	0.200	1.207
433	-	-	1.297	0.000	1.252	0.000	1.234
430	1.13	-	-	-	-	-	-
400	0.214	-	(0.29)	-	(0.281)	-	(0.271)

Fundamental sensations determined in this manner are exactly identical with the colors designated by Mr. Hering, based on a purely psychological analysis of sensations of color, as "primary red" [Ur-Rot], "primary green" and "primary blue." The spectral light complementary to fundamental sensation **B**, with an approximate wavelength of 575 μμ, is Mr. Hering's "primary yellow" and corresponds to the crossover wavelength of fundamental sensation curves **R** and **G**.[25]

The anomalous trichromatic systems investigated by us result when maintaining the quality of fundamental sensation **G** but considering the form of its intensity curve to be more similar to that of **R**. If it is changed to a degree where it completely coincides with **R** *only two* hues will be present in the spectrum (albeit in different saturation), namely blue (λ = ca. 470 μμ) and yellow (λ = ca. 575 μμ). If one assumes fundamental sensation $\mathbf{W}_1$ to be equal to yellow and $\mathbf{K}_1$ equal to blue the dichromatic system, hypothetically generated in this manner, is completely identical with the first type of this kind of systems investigated by us. This is actually the case as Messrs. Hippel[26] and Holmgren's[27] investigations of an individual with a dichromatic right and a trichromatic left eye show. The mentioned idea of unchanged quality accompanying changed intensity distribution of fundamental sensation **G** has shown itself to be in complete agreement with experience.

To what extent the other cases of unilateral "color blindness" detected and investigated by Mr. Holmgren can be used to support the doctrine of change in fundamental sensation curves while maintaining the quality of sensation can only be judged with certainty after transitional, so far unknown forms have been found in other groups of anomalous trichromatic systems.

References and notes

1. F.C. Donders, *Gräfe's Archiv* **27** (1), p. 176, 1881.
2. All color equations described below have been determined using a Helmholtz color mixture apparatus modified for quantitative investigations (see *Bericht über die wissenschaftlichen Instrumente auf der Berliner Gewerbeausstellung im Jahre 1879*, Berlin 1880, p.520, and R. Schelske in *Wied. Ann* **16**, p. 349, 1882). Description of the experimental details of determining the color equations requires an amount of detail that cannot be adequately covered in the available space.
3. The owner of the monochromatic color system investigated by us is Trade School Director (ret.) Dr. A Beyssell whose visual sense indicates all properties connected with such anomaly (see F.C. Donders, *Gräfe's Archiv* **30** (1), p. 80, 1884).
4. The reduction coefficients for the conversion to the interference spectrum were determined from the index of refraction of the prism in use, those for the conversion to sun light with the help of a special photometric measurement (see A. König, *Verhandl. d. Physikal. Gesellsch. In Berlin* May 22, 1885 and March 29, 1886). T*he term sun light is always used for light diffusely reflected by a plane covered with magnesium oxide and directly illuminated by solar rays from a clear sky.* See A. König, *Gräfe's Archiv* **30** (2), p. 162, 1884 and *Wied. Ann* **22**, p. 572, 1884.

5. F.C. Donders, New researches on the systems of coloursense. *Onderzoek gedaan in het Physiolog. Laboratorium der Utrecht'sche Hoogeschool*, 3de Reeke D. VII, Bl. 95, 1882 and *Gräfe's Archiv* **30** (1), p. 15, 1884.
6. This is in principle the same method as used, at the recommendation of Mr. Donders, by Mr. van der Weyde in case of dichromatic systems. – See F.C. Donders, *Proces-verbal der K. Akad. von Wetenschappen, Amsterdam*, Afd. Natuurkunde, Zitting van 26. Febr., 1881. – F.C. Donders, *Gräfe's Archiv* **27** (1), p. 155, 1881. – J.A. van der Weyde, Methodisch ondezoek der Kleurestelsels van kleurblinden, *Onderzoekingen gedaan in het Physiol. Labor. der Utrecht'schen Hoogeschool*, 3de Reeks, D. VII, Bl.1, 1881. J.A. van der Weyde, *Gräfe's Archiv* **28** (1), p. 1, 1882.
7. A so far not accurately determined dependence appeared only when the color equations contained spectral light strongly absorbed by the pigment of the macula lutea. To the extent possible its influence was eliminated by using the light employed in different mixtures in this part of the spectrum, when feasible, in identical intensities. – Mention needs to be made here that in a fifth investigated dichromatic system the presence of intensity independence of this kind could not be confirmed with certainty also in other parts of the spectrum. This system was not further considered for this article because its owner, a young physicist, himself plans to investigate it in more detail but he has so far not found the necessary time.
8. Complete agreement cannot be expected because in gas as well as sun light this experimentally determined location (the "neutral point") moves with increasing intensity in the direction of the blue end of the spectrum. Resolution of the controversy between Mr. E. Hering and one of us (K.) regarding the location of the intensity dependent neutral point must be deferred to another place and time.
9. The persons endowed with these color systems are Privy Councillor Prof. W. Waldeyer, Mr. cand. phil. E. Brodhun, Assessor L. Kranke and Dr. med. H. Sakaky.
10. A. König, *Gräfe's Archiv* **30** (2), p. 165, 1884 and *Wied. Ann* **22**, p. 567, 1884.
11. Mr. van der Weyde has used as a light source a matte sheet of glass inserted into a window frame. Under the conditions mentioned by him it probably resulted in a somewhat more bluish hue than our magnesium oxide directly illuminated by sunlight.
12. This comment only relates to congenital "color blindness." Other forms exist among pathologically developed anomalies.
13. The results in this respect reported by J.J. Müller (*Gräfe's Archiv* **15** (2), p. 208, 1869) are not in agreement with our results and those of *all other* investigators.
14. Rayleigh, *Nature* **25**, p. 64, 1881. (Read before Section A of the British Association, 2. Sept. 1881.)
15. F.C. Donders, *Onderzoek*, etc. 3de Reeks, D. VIII, Bl. 170 and *du Bois Reymond's Archiv für Physiol*. Vol. 1884, p. 518.
16. Among 70 investigated trichromatic systems we have only found 3 belonging to this group.
17. J. Fraunhofer, *Denkschriften der Bayerischen Akademie*, 1815.
18. J. Cl. Maxwell, On the theory of compound colours, *Phil. Trans. of the R. Soc. of London* **150** (1), p. 57, 1860.
19. F.C. Donders, New researches on the systems of coloursense, *Onderzoekingen gedaan in het Physiol. Labor. der Utrecht'schen Hoogeschool*, 3de Reeks, D. VII, Bl. 95, 1882.

20. The characteristic indicator advantageously chosen by Mr. Donders for the purpose of distinguishing among trichromatic systems is the ratio in which lithium red (λ = 670 μμ) and thallium green (λ = 535 μμ) must be mixed to obtain the hue of sodium yellow (λ = 590 μμ). We obtained for the more closely investigated trichromatic systems the following ratios:

	Li : Tl
for K.	2.66:1
for D.	3.25:1
for B.	0.71:1
for Z.	0.96:1.

21. The meaning of the term fundamental sensations of color is completely identical with the meaning of Mr. Donders' earlier mentioned term fundamental color.
22. F.C. Donders, *Gräfe's Archiv* **30** (1), p. 15, 1884.
23. It is possible to select in the expressions for $\mathbf{W}_2$ and **G** the values for the coefficients β''_2 and c'' to be different from zero without disturbing the relationships so far found. But because the observational data presently in our possession do not provide a good reason for doing so we have made the simplest assumption and equated $\beta''_2 = c'' = 0$. In addition, the uncertainty of observation did not make it possible to decide if one should, in order to obtain even better agreement, give the superposition coefficients α''_1, α''_2, a''' and b''' a small value different from zero rather than zero as used by us.
24. The above-mentioned significant uncertainty of curve *R*' does not call the validity of this claim into question but only that of the numerical value of the superposition coefficients.
25. This is the same position in the spectrum where, according to the investigations conducted on 111 eyes by Mr. Donders (*du Bois Reymond's Archiv für Physiol*. Vol. 1884, p. 535), most people place the purest yellow.
26. A. v. Hippel, *Gräfe's Archiv* **26** (2), p. 176, 1880 and 27 (3) p. 47, 1881.
27. F. Holmgren, *Centralblatt f. d. med. Wissenschaften* 1880, p. 898. – Congrès internat. périodique des sciences médicales. 8. Session, Copenhague 1884, Section d'Opthalmologie, -- *Ann. d'Oculistique* **92**, p. 132, 1884.

[Note: Text emphases by the author, indicated in the original manuscript by horizontal double spacing, have been expressed in italic in the translation. German fracture font used as labels for fundamental sensations in the text and figures of the original has been replaced in the text with bold font.]